PROCESSES OF ABSTRACTION IN CONTEXT THE NESTED EPISTEMIC ACTIONS MODEL

Tommy Dreyfus, Tel Aviv University, Israel

The nested epistemic actions model for abstraction in context has been developed with the purpose of analyzing processes of mathematical abstraction in context. We take a point of view influenced by activity theory and consider situations in which the learners' motivation for a new construct, as well as their thinking mode, are mathematical. Abstraction is then defined as an activity of vertically reorganizing previous constructs that leads to the emergence of a new construct. Processes are considered as they occur along a sequence of activities in a learning environment with a curricular, social and historical context. The basic elements of the model are three epistemic actions: Recognizing, Building-with and Constructing. These epistemic actions are nested within each other. Different constructing actions may go on in parallel and interact with each other. The patterns of interaction may indicate events such as the justification of a mathematical relationship by the learner or learner's relationship to contextual elements such as technological tools. The consolidation of new constructs along a sequence of activities forms an integral part of the model. Constructing and consolidating processes are considered in a variety of social contexts ranging from solitary learners to entire classrooms.

This paper is about an empirically based theoretical model we have used for the past six years in order to analyze processes of abstraction in context. The paper begins with an overview, in which I attempt to organize the central ideas into a diagram, and continues with an example-based discussion of the most important ideas.

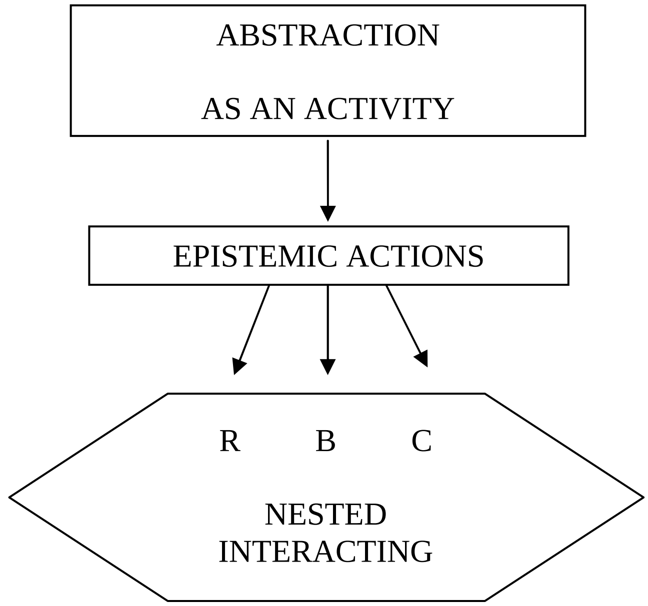
INTRODUCTION AND OVERVIEW

Processes of learning and teaching mathematics are complex in any setting, and especially so in a classroom. In order to somewhat reduce the complexity, the focus of this talk will be mainly on *learning* and *mathematics*, and on the *processes* by which this may happen. In other words, the cognitive aspects of the learning processes will form the center of attention. But even though the main focus will be on the cognitive aspects, the learning process will be considered within the complex context in which it occurs.

Taking a constructivist perspective, we are interested in the emergence of new (for the learner) constructs. More specifically, we adopt the Freudenthal School's idea of vertical mathematization and consider how new mathematical constructs emerge by vertically reorganizing previous constructs and establishing connections between them. We will use the term *abstraction* (or abstracting) to denote such reorganization.

Many contextual factors influence how such a process of abstraction may be realized in a specific case. This *context* has several components: A curricular context, often including a sequence of activities designed with specific learning goals in mind; a historical context, including the students' previous learning experiences; a learning context, possibly including computerized environments with technological tools that may be at the students' disposal; and a social context, which may be an alternation of group work, individual work and whole class work. We thus speak about abstraction in context.

As researchers, we need methodological means to investigate processes of abstraction in context, and these means must be adapted to our goals. Our goal is a fine-grained analysis of abstraction in context. Based on our research, teaching and development activity over many years, we have been led to adopt an *activity* theory point of view (Leont'ev, 1981; Kutti, 1996). We will thus consider abstraction as an activity composed of *actions*. This enables us to look at the process aspects of abstraction, and implies that the meaning of the learners' actions is inseparable from their goals, and must be seen within the activity, in which their overall *motives* drive their actions.



The most crucial actions in processes of abstraction tend to be mental actions, which are not directly observable. The idea of *epistemic actions* helped us overcome this obstacle: Epistemic actions are mental actions made observable via students' verbalizations or physical actions. On the basis of prior empirical research, we identified three epistemic actions that are relevant for processes of abstraction: *Recognizing* (abbreviated as R), *Building-with* (B) *and Constructing* (C). The model (called temporarily *RBC-model*) specifies how these actions are influenced by context, and how they interact and merge in processes of abstraction.

The model was used by our research group as well as by independent researchers using a variety of data. This led not only to validation of the model, but also to its further development and refinement. We found that the epistemic actions occur *nested* in specific ways; that *constructions* may go on in *parallel*, branch, combine and *interact* in other ways; and that in a sequence of activities, *consolidation* of a learner's new constructs may occur during further constructions. With respect to social context, we found that social-interaction episodes and cognitive episodes evolve in parallel; we found that the manner in which the learner interacts with computerized environments determines whether constructions branch or combine; and with respect to curricular context, we found, maybe obviously, that task design is crucial in enabling the emergence of understandings.

In the sequel, I will discuss in more detail, and with examples, abstraction as an activity, the RBC epistemic actions, nesting and interacting parallel constructions, consolidation, and the role of various contextual aspects.

Before going on, though, two remarks are in order

- While this presentation is primarily theoretical, our ideas, including the definition of abstraction and the decision to use epistemic actions are rooted in the pedagogical experience of team members as teachers and curriculum developers, as well as in a long sequence of empirical studies. The theoretical ideas have grown over several years through our discussion of fine-grained data of processes of learning.

- Since this paper is being prepared for a lecture, it has a single author. However, the research program and the model for abstraction have been initiated in common by Rina Hershkowitz, Baruch Schwarz and Tommy Dreyfus, and further developed by eight team members and at least eight additional researchers independent of the team, who have adopted the model for their research, and in some cases proposed to modify it. This explains the frequent use of the pronoun 'we' when referring to the developers of the model.

ABSTRACTION AS AN ACTIVITY

In order to show what we mean by abstraction as an activity, let's consider the following example (for more details, see Dreyfus, Hershkowitz and Schwarz, 2001). Beginning algebra students (grade 7), working in pairs, were presented with several numerical examples of 2 by 2 arrays of integers (which were called "seals"), all of the form indicated in the accompanying figure. After listing a variety of properties of the seals, the students' attention was focused on the diagonal product property, namely that the difference between the products of the elements in the two diagonals is 12. The students were asked whether they thought this property was true for all such seals. The activity was set up so that the diagonal product property appeared surprising and this motivated the students to justify their answer. They had just started learning how to represent a variable by a letter, and had used the regular distributive law a(c+d)=ac+ad, but not the extended distributive law (a+b)(c+d)=ac+ad+bc+bd. Nor had they, to the best of our knowledge, ever used algebra as a tool for justification. Nevertheless, the task was presented to the students in a context where it was natural for them to use algebra. The task thus presented them with an opportunity to use, for the first time, algebra as a tool for justification and thus with an opportunity for constructing the idea that algebra is a tool for justification. Furthermore, using algebra to show the diagonal product property requires using the extended distributive law. The activity thus presented the students also with the need and with an opportunity for constructing this law.

x	x+6
x+2	x+8

We note that the task was adapted to the student population, as well as to their prior learning history as a group. We further note that the entire development and thinking mode is mathematical, and that the motivation for constructing new mathematical ideas and procedures arises out of the mathematics, rather than from some external motivation. Specifically, the distributive law appears as a mathematical tool needed to solve the justification problem; it is an answer to a mathematical need.

Constructs like the idea of algebra as a tool for justification or the extended distributive law that are new to the students and arise in mathematical situations such as above, closely correspond to the notion of vertical mathematization as posited and used by the Freudenthal school (e.g., Treffers and Goffree, 1985). Vertical mathematization is the process of constructing a new mathematical construct within the mathematics itself and by mathematical means; it typically requires the reorganization of previous constructs, establishing relationships and connections between them, interweaving them into a single process of mathematical thinking, with the purpose of arriving at a new mathematical construct.

We contend that the process of such vertical mathematization together with the ensuing emergence of new mathematical constructs merits to be called a process of abstraction. We thus *define abstraction as an activity of vertically reorganising previous mathematical constructs into a new mathematical construct*.

According to this definition, abstraction is not an objective, universal process but depends strongly on context, including the history of the students participating in the activity of abstraction and the artefacts available to them. The artefacts include material objects and tools, such as computerised ones, as well as immaterial ones including language and procedures that are often outcomes of previous activities (knowledge artefacts).

While we are not specialists in other content domains, it may be that the multi-layered structured nature of the content domain of mathematics makes the above definition of abstraction more appropriate for mathematics than for other content domains. Many other mathematical constructs can be seen as arising from vertical reorganization; they include establishing connections between two representations for the same concept (e.g., Schwarz and Dreyfus, 1995), building a procedure that includes as special case two (or more) previously used procedures (reference), realizing that a concept such as rate, or fraction can be looked at from many different points of view (e.g., Thompson and Thompson, 1996; Thompson and Saldanha, 2003), or justifying a statement or process that has been known and used for a long time (reference).

On the other hand, there is mathematical learning which does not require, nor usually lead to vertical reorganization, such as the building of algorithmic, procedural knowledge in a mechanical way or rules that are remembered rather than constructed and can therefore not be reconstructed. More generally, learning that aims at routinization, though we don't deny its potential usefulness and necessity, is not within the scope of the theoretical framework we present here.

THE RBC EPISTEMIC ACTIONS AND OBSERVABILITY

The description in the previous section leaves the question open how abstraction in context happens. How can the processes such as the first emergence of the idea of the construct "algebra as a tool for justification" be described and analyzed at a fine-grained micro-level? And if data for subsequent activities are available, in which the construct becomes more familiar to the students, how can this process of *consolidation* be defined, described and analyzed. And how can one analyze the interactions and influences of the context, in which these processes occur?

As observed above, we are taking an activity theory point of view. We are thus looking to describe students' thinking in terms of actions. The class of actions relevant for this purpose are epistemic actions, which have been defined as actions used in processes of constructions of knowledge (Pontecorvo & Girardet, 1993; Schwarz & Hershkowitz, 1995). Epistemic actions are observable in the sense that students' verbalizations or non-verbal actions attest to them. Our attempts to analyze processes of abstraction, the ones in the diagonal product property example above as well as others related to rate of change as a function (Hershkowitz, Schwarz and Dreyfus, 2001), have led us to decide on the three specific epistemic actions of Recognizing, Building-with and Constructing. We make no claim that these epistemic actions are unique. Other epistemic actions have been shown to be useful in history learning (Pontecorvo & Girardet, 1993). But these three epistemic actions did emerge from our data, and their usefulness was later validated by use on a variety of other data. Some of these data were collected by us, and some by other researchers. Some of the data collected by us were collected for the specific purpose of being analyzed by means of the RBC-model, and others were available to us after having been collected for different purposes.

Constructing is the central step of abstraction. It consists of assembling or integrating elements of one's knowledge to produce a new construct; for example, the students described above constructed the extended distributive law by using elements of algebra, including the regular distributive law, which they had constructed earlier and recognized as being useful in the present task. *Recognizing* a mathematical construct, which is familiar from a previous activity, occurs when a student realizes that this construct is inherent in, connected to or relevant for the mathematical situation in the present activity. Recognising may occur in at least two ways, by analogy and by specialisation. *Building-with* consists of combining existing knowledge elements in order to meet a goal such as solving a problem or justifying a statement; for example, the extended distributive law was arrived at by building-with the regular distributive law and other elements that had been recognized by the students as relevant. It is important to realize that none of the student pair we analyzed arrived at the extended distributive law in the same way a mathematician might derive the extended law from the regular one. In the students' process, false starts, dead ends and intuitive guesses were involved, and each pair had their own way to achieve the goal (or fail to achieve it).

The same task may lead to building-with by one student but to constructing by another, depending on the student's personal history, and more specifically on whether or not the required constructs, such as the extended distributive law, are already at the student's disposal. Another important difference between constructing and building-with lies in the relationship of the action to the motive driving the activity: In building-with constructs, the goal is attained by using knowledge that was previously acquired or constructed. In constructing, the process itself, namely the construction or restructuring of knowledge is often the goal of the activity; and even if it is not, then it is at least indispensable for attaining the goal. The goals students have (or are given) thus strongly influence whether they build-with or construct. If they solve a standard problem, they are likely to recognise and build-with previously acquired structures. If they solve a non-standard problem, they might be faced with an obstacle that causes them to construct by vertically reorganising their knowledge to overcome the obstacle.

NESTING AND INTERACTING PARALLEL CONSTRUCTIONS

Our first formulation of the RBC-model included the view that the three epistemic actions are the basic elements of a model, called the dynamically nested RBC model of abstraction. According to this model, constructing incorporates the other two epistemic actions in such a way that building-with actions are nested in constructing actions and recognising actions are nested in building-with actions and in constructing actions. Further research using the model led to considerable enhancement and refinement of the model. For example, the task given as example above made us realize that constructing actions (the extended distributive law) can be nested in other, higher level constructing actions (algebra as a tool for justification). The complexity of the mathematical topic and tasks under consideration has a lot to do with the additional nesting.

In further research, we observed the emergence of constructs in more advanced mathematics, namely bifurcations in dynamic systems, and more specifically we observed how a solitary learner, a rather sophisticated mathematician, constructed a justification for the second bifurcation, leading from two to four branches in the logistic dynamic process (Dreyfus and Kidron, 2006). This process of constructing was similar to the one by the grade 7 students mentioned above, in that the process was driven by a similar motive: the need for justification. However, it was different in its complexity. The complexity of the task led the learner to use elements from algebra, from analysis and from numerical analysis; she used various representations including verbal and graphical ones, the graphical ones being static and dynamic. Our analysis showed that the complexity of the topic and the complexity of the task led to a corresponding complexity of constructing actions, the main one being composed of four secondary ones, which interacted with each other in various ways. These interactions included a constructing action branching off from an ongoing one, an ongoing constructing action being interrupted (in favor of another one) and being resumed later, and constructing actions, which had branched at an earlier stage recombining at a later stage. The analysis further showed that the combining of constructions leads to enlightenment, an insight into the statement and in this sense to a justification in the eyes of the learner (Kidron, 2006). We note that these refinements and further developments of the RBC-model are empirically based in the sense that they arise from the use of the model for analyzing data.

CONSOLIDATION

Sequences of activities are particularly important in mathematics learning since constructs are typically multifaceted compounds of previous constructs, and thus reference to previous constructs in further activities is indispensable in processes of abstraction. It should be expected that previous constructs are being recognized and built-with with increasing ease as they keep being used.

Several research studies based on the RBC-model have treated consolidating as a process that follows constructing somewhat independently (Dreyfus, & Tsamir, 2004; Monaghan, & Ozmantar, 2004, in press; Tabach, & Hershkowitz, 2002; Tabach, Hershkowitz, & Schwarz, 2006). For example, Dreyfus and Tsamir (2004) have developed an empirically based, theoretical analysis of consolidation that emerges from a sequence of interviews about the comparison of infinite sets with a talented student. Their analysis shows that consolidation can be identified by means of the psychological and cognitive characteristics of self-evidence, confidence, immediacy, flexibility and awareness. They also found that modes of thinking related to problem solving and to reflective activity are conducive to consolidation.

Dreyfus, Hadas, Hershkowitz and Schwarz (2006), on the other hand, argued that constructing processes and consolidating processes are often narrowly intertwined, and presented data in support of this argument. Processes of abstraction of a group of students working together in a classroom on tasks from a unit on probability were analyzed with the aim of identifying mechanisms for consolidating recent knowledge constructs. Three such mechanisms were identified by means of indicative epistemic actions: Consolidating during building-with the construct, consolidating during reflecting on the construct, and consolidating by recognizing the construct as relevant for constructing further constructs during a new process of abstracting. The third mechanism is particularly interesting because of the strong linkage it establishes between the processes of constructing and consolidating. Given the centrality of consolidation for learning processes, the model will henceforth be called the *RBC+C model*, the second C standing for Consolidation.

CONTEXTS

Curricular

As pointed out above, task designs using conflicts, surprise and uncertainty (see, e.g. Hadas, Hershkowitz and Schwarz, 2000) are conducive to intra-mathematical motivation. If such a design creates the need for a new construct, it is likely to be conducive to a process of abstraction. The question whether or not the need for a new construct arises depends of the learners' previous experience, and therefore, within a curriculum, on the sequencing of tasks.

Processes of abstraction and consolidation have been investigated within a task sequence with the purpose to introduce 8th grade students to basic notions of probability such as experimental versus theoretical probability (an appropriate form of the law of large numbers), and one- and two dimensional sample spaces. Specifically, two dimensional sample spaces were limited to cases of either binary cases (only two events per dimension) or equal probability events. The sequence lasted for ten lessons. Data were collected in from six classrooms and fourteen groups of students (eight groups in the classrooms and six in laboratory conditions). This research project allowed us to thoroughly investigate consolidation and led to the discovery of the mechanisms pointed out above. The project also led to detailed results about the flow of knowledge constructing by individuals from one individual to the other within an interacting group of learners (Hershkowitz, Hadas, Dreyfus and Schwarz, in press; see also Hershkowitz, 2007).

A further discovery of this project concerns the importance of partially correct construct (PaCCs) for learning processes (Ron, Dreyfus and Hershkowitz, 2006). PaCCs can illuminate phenomena such as misconceptions or alternative conceptions at a micro-level. They have been used to explain how students' incorrect answers sometimes overshadow meaningful knowledge they have constructed and, on the other hand, how correct answers often hide knowledge gaps. They thus explain, for example, how students can successfully accomplish a sequence of several tasks without apparent difficulty and later run into difficulty, when working on a task, which seems to require only actions they have previously carried out well. In order to explain this difficulty, the students' learning process during part of the probability task sequence is analyzed, and PaCCs were identified. Several different types of PaCCs were found.

Technological tools

The variety of technological tools used in mathematics education is very wide, and our experience is not sufficiently broad yet to make any general statements. Therefore we concentrate here on a specific case, namely the same research discussed already in the section on nesting and interacting parallel constructions. The solitary learner struggling to justify the transition form 2-period to 4-period in the logistic dynamic process intensively used a computer algebra system to support her investigation. Our research showed that in phases in which the computer was used for open exploration and thus software often determined where the learner was turning next, constructing processes tended to branch into two; on the other hand, when the learner controlled the tool, the tool efficiently facilitated the combining of constructions and thus lead to added connectivity and the learner's enlightenment and her justification of the mathematical phenomenon (Kidron and Dreyfus, in preparation).

From the methodological point of view, two issues are important here. First, the data about this learning process, like several others, were not collected for the purpose of being analyzed by the RBC+C model. Second, the branching and combining of constructions were identified on the basis of the epistemic actions, long before we identified the relationship between them and learner's use of the tool, as well as the significance of combining constructions for justification. This shows once again the dual complementary nature of the model as methodological tool and as theory in development.

Social

The context with which we began our program concerned an individual student who solved a mathematical problem by himself (Hershkowitz, Schwarz, & Dreyfus, 2001). Such a context is problematic for observing epistemic actions. Therefore, an experimenter was present to ask the student to think aloud while solving the problem. We initiated our research program with this "problematic" context for two reasons: We believe in the importance of the single individual’s mathematical thinking, autonomy, in abstraction; and the methodological “loss” is also a methodological “gain” since it avoids the issue of distributed (construction of) knowledge, and thus makes the identification of epistemic actions simpler.

The research based on the seals activity mentioned above, was the first one, in which we set out to investigate the role of social context in processes of abstraction within the RBC framework. By the way, the data used in this research were available from before we had first thought of the model and were not collected specifically. Student pairs collaboratively solved the seals problem in the presence of a moderator. This context reflected the fact that we were interested in abstraction emerging from collaborative practices. Of course this interest does not conflict with the goal of autonomy which is central in abstraction: in dyadic interaction, each individual is expected to participate substantively in the abstraction activity. The problem is then methodological: how to cope with the identification of epistemic actions during discussions. This is what we did in a study on two pairs of students who solved a problem presented as an arithmetical issue but that demanded them to elaborate algebraic considerations and rules (Dreyfus, Hershkowitz, & Schwarz, 2001). Two parallel analyses of the protocols of the work of the student pairs were carried out, an analysis of the epistemic actions of abstraction as well as an analysis of the peer interaction. These parallel analyses led to the identification of types of social interaction that support processes of abstraction. We identified several patterns of interaction that indicated the kind of participation of each peer in abstraction.

Finally, maybe the most relevant of our investigations of the role of social context in processes of abstraction is the investigation of such processes as they occur in groups working within a classroom. As pointed out above, research falling in this category will be discussed by Hershkowitz (2007) at this workshop.

CONCLUSION

In summary, the RBC+C model of abstraction in context considers that the genesis of an abstraction passes through three stages: a need for a new construct, its emergence and its consolidation. As a methodological tool, the model takes into account a variety of aspects of the process of abstraction: The emergence of mathematical constructs by mathematization leading to vertical reorganization; the consolidation of these new constructs in the course of a sequence of linked activities; learning in different collaborative and individual social settings, including classrooms in which students move between working alone, group work and whole class discussion, but also solitary learners; and technological tools supporting the process of abstraction. The model gives tools to carry out a fine-grained analysis of complex processes of abstraction taking into account these contextual factors. Moreover, recent research has shown the analytical explanatory power of the model, for example establishing a link between the combining of constructions and justification. Finally, we expect that the model will serve as a tool to provide additional structure and insight to other theories. For example, work is under way to analyze the transition from model-of informal mathematical activity to model-for more formal mathematical reasoning (Gravemeijer, 1999) as a process of abstraction, in the case of the comparison of fractions (Weiss and Dreyfus, in preparation). This closes a circle by leading us back to the Freudenthal School's ideas and contribution, from which this lecture and our research program started.

REFERENCES

Dreyfus, T., Hadas, N., Hershkowitz R. and Schwarz B. B. (2006). Mechanisms for consolidating knowledge constructs. In J. Novotná, H. Moraová, M. Krátká, & N. Stehliková (eds.), *Proceedings of the 30th Conference of the International Group for Psychology of Mathematics Education*, Vol. 2. Prague, Czech Republic: Charles University Faculty of Education, pp. 465-472.

Dreyfus, T., Hershkowitz, R. and Schwarz, B. (2001). Abstraction in Context II: The case of peer interaction. *Cognitive Science Quarterly 1*(3/4), 307-368.

Dreyfus, T. and Kidron, I.: Interacting parallel constructions: A solitary learner and the bifurcation diagram. Submitted for publication to *Recherches en didactique des mathématiques 25*,?? .

Dreyfus, T. and Tsamir, P. (2004). Ben's consolidation of knowledge structures about infinite sets. *Journal of Mathematical Behavior 23* (3), 271-300.

Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning.1* (2), 155-177.

Hadas, N., Hershkowitz, R. and Schwarz, B. (2000). The role of contradiction and uncertainty in promoting the need to prove in dynamic geometry environments. *Educational Studies in Mathematics, 44* (1-2), 127-150.

Hershkowitz, R. (2007). Contour lines between a model as a theoretical framework and the same model as methodological tool. This workshop.

Hershkowitz, R., Hadas, N., Dreyfus, T. and Schwarz, B. (in press). Processes of abstraction from the diversity of individuals' constructing of knowledge to a group's "shared knowledge". Accepted for publication by *Mathematics Education Research Journal.*

Hershkowitz, R., Schwarz, B. B. and Dreyfus, T. (2001). Abstraction in context: Epistemic actions. *Journal for Research in Mathematics Education 32* (2), 195-222.

Kidron, I. & Dreyfus, T. (2006). Mathematical justification, parallel constructions of knowledge and the instrumentation process. *Proceedings of the 9th Conference on Research in Undergraduate Mathematics Education*. Piscataway, NJ.

Kidron, I. & Dreyfus, T. (in preparation). A computer algebra system and mathematical justification in parallel interacting constructions.

Kuutti, K. (1996). Activity theory as a potential framework for human-computer interaction research. In B. A. Nardi (Ed.), *Context and consciousness: activity theory and human-computer interaction* (pp. 17-44). Cambridge, MA: MIT Press.

Leont'ev, A. N. (1981). The problem of activity in psychology. In J. V. Wertsch (Ed. and Trans.), *The concept of activity in Soviet psychology* (pp. 37-71). Armonk, NY: Sharpe.

Monaghan, J., and Ozmantar, M.F. (2006). Abstraction and consolidation. *Educational Studies in Mathematics* 62, 233-258.

Pontecorvo, C. and Girardet, H. (1993). Arguing and reasoning in understanding historical topics. *Cognition and Instruction 11*, 365-395.

Ron, G., Dreyfus, T., & Hershkowitz, R. (2006). Partial knowledge constructs. In J. Novotná, H. Moraová, M. Krátká, & N. Stehliková (eds.), *Proceedings of the 30th Conference of the International Group for Psychology of Mathematics Education*, Vol. 4. Prague, Czech Republic: Charles University Faculty of Education, pp. 449-456.

Schwarz, B. B. and Dreyfus, T. (1995). New actions upon old objects: A new ontological perspective on functions. *Educational Studies in Mathematics 29*, 259-291.

Schwarz, B. B. and Hershkowitz, R. (1995). Argumentation and reasoning in a technology-based class. In J. D. Moore and J. F. Lehman (Eds.), *Proceedings of the Seventeenth Annual Meeting of the Cognitive Science Society*. Mahwah, NJ: Lawrence Erlbaum Associates, pp. 731-735.

Tabach, M. and Hershkowitz, R. (2002). Construction of knowledge and its consolidation. In A. D. Cockburn, & E. Nardi (Eds.), *Proceedings of the 26th International Conference for the Psychology of Mathematics Education*, Vol. 4. Norwich, UK: UEA, pp. 265-272.

Tabach, M., Hershkowitz, R. and Schwarz, B. B. (2006). Constructing and consolidating of algebraic knowledge within dyadic processes: A case study. *Educational Studies in Mathematics* 63, 235-258.

Thompson, P. W. and Saldanha, L. A. (2003). Fractions and multiplicative reasoning. In J. Kilpatrick, W.G. Martin & D. Shifter (Eds.), *A research companion to principles and standards for school mathematics*. Reston, VA: NCTM, pp. 95-113.

Thompson, A. G. and Thompson, P. W. (1996). Talking about rates conceptually, Part II: Mathematical knowledge for teaching. *Journal for Research in Mathematics Education, 27*(1), 2-24.

Treffers, A. and Goffree, F. (1985). Rational analysis of realistic mathematics education – The Wiskobas program. In L. Streefland (Ed.), *Proceedings of the 9th International Conference for the Psychology of Mathematics Education* (Vol. 2, pp. 97-121). Utrecht, The Netherlands: OW&OC.

Weiss, D. and Dreyfus, T. (in preparation). The transition from model-of to model-for as a process of abstraction: The case of the comparison of fractions.